

Proof of a conjecture on ‘plateaux’ phenomenon of graph Laplacian eigenvalues

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Abstract

Let G be a simple graph. A pendant path of G is a path such that one of its end vertices has degree 1, the other end has degree ≥ 3 , and all the internal vertices have degree 2. Let $p_k(G)$ be the number of pendant paths of length k of G , and $q_k(G)$ be the number of vertices with degree ≥ 3 which are an end vertex of some pendant paths of length k . Motivated by the problem of characterizing dendritic trees, N. Saito and E. Woei conjectured that any graph G has some Laplacian eigenvalue with multiplicity at least $p_k(G) - q_k(G)$. We prove a more general result for both Laplacian and signless Laplacian eigenvalues from which the conjecture follows.

Keywords: Laplacian eigenvalue, Pendant path, Signless Laplacian eigenvalue

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1 Introduction

Let G be a simple graph. A *pendant path* of G is a path such that one of its end vertices has degree 1, the other end has degree ≥ 3 , and all the internal vertices have degree 2. Let $p_k(G)$ denote the number of pendant paths of length k of G , and $q_k(G)$ denote the number of vertices with degree ≥ 3 which are an end vertex of some pendant paths of length k . Saito and Woei [4] studied Laplacian eigenvalues of dendritic trees. They observed eigenvalue(s) ‘plateaux’ (i.e.,

set of eigenvalue(s) with multiplicity) in the Laplacian eigenvalues of such trees. More generally, they showed that $(3 \pm \sqrt{5})/2$ is a Laplacian eigenvalue of any graph G with multiplicity at least $p_2(G) - q_2(G)$. This motivated them to put forward the following conjecture.

Conjecture 1. ([4]) *For any positive integer k , any graph G has some Laplacian eigenvalue with multiplicity at least $p_k(G) - q_k(G)$.*

We remark that the special cases $k = 1$ follows from a result of [2] (see also [3]) asserting that multiplicity of 1 as a Laplacian eigenvalue of a graph G is at least $p_1(G) - q_1(G)$. We prove a more general result (Theorem 4 below) for both Laplacian and signless Laplacian eigenvalues from which Conjecture 1 follows.

2 Preliminaries

Let G be a simple graph with vertex set $\{v_1, \dots, v_n\}$ and edge set $\{e_1, \dots, e_m\}$. The *adjacency matrix* of G is an $n \times n$ matrix $A = A(G)$ whose (i, j) -entry is 1 if v_i is adjacent to v_j and 0 otherwise. The *incident matrix* of G , $X = X(G) = (x_{ij})$, is an $n \times m$ matrix whose rows and columns are indexed by vertex set and edge set of G , respectively, where

$$x_{ij} = \begin{cases} 1 & \text{if } e_j \text{ is incident with } v_i, \\ 0 & \text{otherwise.} \end{cases}$$

If we orient the edges of G , we may define similarly the *directed incidence matrix* $D = D(G) = (d_{ij})$, with respect to the particular orientation, as

$$d_{ij} = \begin{cases} +1 & \text{if } e_j \text{ is an incoming edge to } v_i, \\ -1 & \text{if } e_j \text{ is an outgoing edge from } v_i, \\ 0 & \text{otherwise.} \end{cases}$$

The matrices $L(G) := DD^\top$ and $Q(G) := XX^\top$ are called the *Laplacian matrix* and *signless Laplacian matrix* of G , respectively. It is easily seen that $L(G) = \Delta - A$ and $Q(G) = \Delta + A$ where Δ is the diagonal matrix of vertex degrees of G .

For any $n \times m$ matrix M , and nonzero real λ , we have

$$\det \begin{pmatrix} \lambda I_n & M \\ M^\top & \lambda I_m \end{pmatrix} = \lambda^{m-n} \det(\lambda^2 I_n - MM^\top).$$

So we come up with the following lemma.

Lemma 2. *For any graph G , $\lambda_1, \dots, \lambda_r$ are all the nonzero eigenvalues of $Q(G)$ (resp. $L(G)$) if and only if $\pm\sqrt{\lambda_1}, \dots, \pm\sqrt{\lambda_r}$ are all the nonzero eigenvalues of $\begin{pmatrix} O & D \\ D^\top & O \end{pmatrix}$ (resp. $\begin{pmatrix} O & X \\ X^\top & O \end{pmatrix}$).*

It is known that, G is bipartite if and only if $L(G)$ and $Q(G)$ are similar (see [1, Prop. 1.3.10]). We will need the following variation of this result.

Lemma 3. *Let G be a graph. Then G is bipartite if and only if with respect to any orientation, any principal submatrix of*

$$\begin{pmatrix} O & D \\ D^\top & O \end{pmatrix} \quad (1)$$

is similar to the corresponding submatrix of

$$\begin{pmatrix} O & X \\ X^\top & O \end{pmatrix}. \quad (2)$$

Proof. If (1) and (2) are similar, then they share the same nonzero eigenvalues, and squaring those eigenvalues, by Lemma 2, lead to the nonzero eigenvalues of $L(G)$ and $Q(G)$. Now [1, Prop. 1.3.10] implies that G is bipartite.

For the converse, suppose that G is bipartite with bipartition (V_1, V_2) . For two different orientations of G , the corresponding matrices (1) are similar. Because switching the orientation of any edge e is equivalent to multiplying the column and the row of (1) corresponding to e by -1 . So we may assume that all the edges are oriented from V_1 to V_2 . Now we get (2) by multiplying (1) from left and right by the ± 1 -diagonal matrix whose -1 entries correspond to the vertices in V_1 . Since (2) and (1) are similar by a diagonal matrix, the corresponding principle submatrices of them are also similar. \square

The *subdivision* of G , denoted by \tilde{G} , is the graph obtained by introducing m new vertices to G , and replacing each edge $e_i = uv$, $i = 1, \dots, m$, by two new edges uw_i and w_iv . It is easily seen that (2) gives $A(\tilde{G})$. Also any orientation of G induces a natural orientation on \tilde{G} , where each $e_i = (u, v)$ is replaced by (u, w_i) and (w_i, v) . We denote this oriented graph by \tilde{G}_o . The matrix (1) can be viewed as the (signed) adjacency matrix of \tilde{G}_o .

3 Main Result

In this section we present the main result of the paper, from which Conjecture 1 follows. We denote the path graph¹ of length k by P_k .

Theorem 4. *Let G be a graph. Then $4\cos^2(\pi i/(2k+1))$ for any $k \geq 1$ and $i = 1, \dots, k$, is both a Laplacian and a signless Laplacian eigenvalue of G with multiplicity at least $p_k(G) - q_k(G)$.*

¹Notice that here the definition of P_k is different from the conventional notation where normally P_k represents the path graph consisting of k vertices.

Proof. For simplicity we fix k , and write p and q for $p_k(G)$ and $q_k(G)$. If $p - q = 0$, there is nothing to prove, so we assume that $p - q \geq 1$. Hence there are p pendant paths of length k which are connected to the rest of the graph via q vertices v_1, \dots, v_q .

We first consider signless Laplacian eigenvalues of G . In view of Lemma 2, it suffices to consider $A(\tilde{G})$. Let $\lambda_1 \geq \dots \geq \lambda_n$ be the eigenvalues of $A(\tilde{G})$. The graph $\tilde{G} \setminus \{v_1, \dots, v_q\}$ contains p connected components isomorphic to P_{2k-1} . We know that the eigenvalues of $A(P_{2k-1})$ consists of $\theta_i := 2 \cos(\pi i / (2k + 1))$ for $i = 1, \dots, 2k$ (see [1, p. 9]). This means that p consecutive eigenvalues of $A(\tilde{G} \setminus \{v_1, \dots, v_q\})$, say $\eta_m, \dots, \eta_{m+p-1}$, are all equal to θ_i . By interlacing (see [1, Corollary 2.5.2]), we have

$$\theta_i = \eta_j \geq \lambda_{j+q} \geq \eta_{j+q} = \theta_i \quad \text{for } j = m, \dots, m + p - q - 1.$$

It follows that θ_i is an eigenvalue of $A(\tilde{G})$ with multiplicity at least $p - q$. So we are done by Lemma 2. Note that $\theta_i = -\theta_{2k+1-i}$.

We now turn to Laplacian eigenvalues. Let G' be the induced subgraph of G by the vertices of these p paths. Note that v_1, \dots, v_q belong to G' . Suppose that the edges of G have been oriented with associated directed incidence matrix $D = D(G)$ and directed subdivision graph \tilde{G}_o . Consider $\tilde{G}_o \setminus \{v_1, \dots, v_q\}$. This graph has at least $p + 1$ connected components, p of which are paths P_{2k-1} with some orientations. Let H be the union of these p directed paths. Let D_1 and X_1 be the submatrices of $D(G')$ and $X(G')$, respectively, obtained by removing the rows corresponding to v_1, \dots, v_q . Then

$$\begin{pmatrix} O & D_1 \\ D_1^\top & O \end{pmatrix} \quad (3)$$

is the (signed) adjacency matrix H . As G' is bipartite, by Lemma 3, (3) is similar to

$$\begin{pmatrix} O & X_1 \\ X_1^\top & O \end{pmatrix} = A(pP_{2k-1}).$$

So each of $\theta_1, \dots, \theta_{2k}$ is an eigenvalue of (3) and so an eigenvalue of the (signed) $A(H)$ with multiplicity at least p . By interlacing, each of $\theta_1, \dots, \theta_{2k}$ is an eigenvalue of

$$\begin{pmatrix} O & D \\ D^\top & O \end{pmatrix}$$

with multiplicity at least $p - q$. Therefore, by Lemma 2, each of $\theta_1^2, \dots, \theta_k^2$ is an eigenvalue of $DD^\top = L(G)$ with multiplicity at least $p - q$. \square

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